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LETTER TO THE EDITOR

On Miura transformations of evolution equations

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Abstract. The general Miura transformation $(t, x, u(t, x)) \to (s, y, v(s, y))$: $v = a(t, x, u, \ldots, \partial^r u/\partial x^r)$, $y = b(t, x, u, \ldots, \partial^r u/\partial x^r)$, $s = c(t, x, u, \ldots, \partial^r u/\partial x^r)$ is considered which connects two evolution equations $u_t = f(t, x, u, \ldots, \partial^n u/\partial x^n)$ and $v_s = g(t, x, u, \ldots, \partial^m u/\partial x^m)$. The conditions $c = c(t)$ and $m = n$ are proven to be necessary. It is shown that every Miura transformation, admitted by a constant separant equation $u_t = f$, consists of the following three transformations: (i) $(t, x, u) \to (t, x, w)$, where $w = \tilde{a}(t, x, u, \ldots, u_{x\ldots x})$; (ii) $(t, x, w) \to (t, y, v)$, where $y = x$ and $v = w$, or $y = w$ and $v = w_x$, or $y = w_x$ and $v = w_{xx}$; (iii) a transformation of time $s = c(t)$ and a contact transformation of (y, v). As an example, the Korteweg–de Vries equation is transformed to three new nonlinear equations, of which two have neither non-trivial algebra of generalized symmetries nor infinite set of conserved densities.

Transformations of nonlinear partial differential equations are very important to modern mathematical physics [2]. Being a generalization of point and contact transformations on one hand and a special case of Bäcklund transformations on the other hand, Miura transformations (MTs), also referred to as equivalence transformations and differential substitutions in [2, 3], have numerous applications, especially to studies of nonlinear evolution equations (EEs). Very often, a newly-found remarkable EE turns out to be nothing but a well studied old EE spoilt by a MT or a chain of MTs [4–7]. MTs make it possible to deduce certain algebraic and analytic properties of a new EE from such properties of the corresponding old EE [2, 8, 9]. A chain of MTs generates a Bäcklund transformation [10–12]. Finally, and this point should be stressed, MTs are applicable not only to completely integrable EEs [3, 9, 12].

Recently, Kingston [13] investigated the structure of the point transformation $(t, x, u(t, x)) \to (s, y, v(s, y))$: $v = a(t, x, u)$, $y = b(t, x, u)$, $s = c(t, x, u)$ between two EEs $u_t = f(t, x, u, u_x, \ldots)$ and $v_s = g(s, y, v, v_y, \ldots)$ and proved the following two theorems: (A) the time transformation must necessarily be of the form $s = c(t)$; (B) if f and g are polynomials in derivatives of u and v, then the condition $b = b(t, x)$ is necessary too. These theorems can simplify calculations essentially. However, Kingston's theorems deal with point transformations only, and one would like to know whether (A) and (B) describe the structure of more general transformations as well. It is easy to see that (B) is not applicable to MTs: the Ibragimov transformation [14] $v = u_x$, $y = u$, $s = t$ connects the polynomial EEs $u_t = u_{xx}$ and $v_s = v^2 v_{yy}$. As for (A), the literature contains no example of a MT with $s \neq c(t)$, and the validity of the theorem for MTs (and for contact transformations as well) will be proven in this letter. We will also prove a theorem which describes the structure of MTs admitted by constant separant EEs. The results will be illustrated by an example.

Let us consider the general MT [2] $(t, x, u(t, x)) \to (s, y, v(s, y))$:

$$v = a(t, x, u, u_1, \ldots, u_r)$$
$$y = b(t, x, u, u_1, \ldots, u_r) \tag{1}$$
$$s = c(t, x, u, u_1, \ldots, u_r)$$

which maps all solutions u of the EE

$$u_t = f(t, x, u, u_1, \ldots, u_n) \tag{2}$$

to solutions v of the corresponding EE

$$v_s = g(s, y, v, v_1, \ldots, v_m) \tag{3}$$

where functions a, b and c are functionally independent, $u_k = \partial^k u/\partial x^k$ and $v_k = \partial^k v/\partial y^k$ $(k = 1, 2, \ldots)$, $\partial f/\partial u_n \neq 0$, $\partial g/\partial v_m \neq 0$, $|\partial a/\partial u_r| + |\partial b/\partial u_r| + |\partial c/\partial u_r| \neq 0$. (Note: t-derivatives of u are not involved in (1) without loss of generality, because u satisfies (2).) As a rule, transformation (1) is not invertible, i.e. u, x and t depend on v, y and s non-locally. The only exceptions are point transformations ($r = 0$ in (1)) and contact transformations ($r = 1$ in (1) under certain restrictions put on a, b and c) [2]. Every EE admits point and contact transformations and generates in this way its own class of mutually equivalent EEs. Since all first-order EEs are mutually equivalent [2], we put $n > 1$ hereafter. It is easy to see that MT (1) connects EEs (2) and (3), i.e. (1) maps all solutions of (2) to solutions of (3), if, and only if, functions a, b, c, f and g satisfy the following condition:

$$\mathcal{D}_s a = g(c, b, a, \mathcal{D}_y a, \ldots, \mathcal{D}_y^m a) \tag{4}$$

where $\mathcal{D}_s = d^{-1}[(\mathcal{D}_x b)\mathcal{D}_t - (\mathcal{D}_t b)\mathcal{D}_x]$, $\mathcal{D}_y = d^{-1}[(\mathcal{D}_t c)\mathcal{D}_x - (\mathcal{D}_x c)\mathcal{D}_t]$, $d = (\mathcal{D}_x b)\mathcal{D}_t c - (\mathcal{D}_x c)\mathcal{D}_t b$, $d \neq 0$ due to functional independence of b and c, $\mathcal{D}_x = \partial_x + \Sigma_{k=0}^{\infty} u_{k+1}\partial_k$, $\mathcal{D}_t = \partial_t + \Sigma_{k=0}^{\infty} (\mathcal{D}_x^k f)\partial_k$, $\partial_x = \partial/\partial x$, $\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial u_k$ $(k = 0, 1, 2, \ldots)$, $u_0 = u$. (Note: (4) must be an identity in $t, x, u, u_1, u_2, \ldots$, because it should not be an ordinary differential equation which restricts solutions u of EE (2).) Analysis of condition (4) will allow us to prove the following.

Theorem 1. If MT (1) connects EEs (2) and (3), and $n > 1$, then $c = c(t)$ and $m = n$ necessarily.

Sketch of proof. Suppose $m \leqslant 1$. Bring (3) to the form $v_s = 0$ by a contact transformation of (s, y, v). Derive from (4) that transformed a and b are functionally dependent (if $n > 1$) in contradiction with independence of the original a, b and c. Suppose $m > 1$. Consider the balance of higher-order derivatives u_k in the left- and right-hand sides of (4). Calculate that $\partial_{r+n}\mathcal{D}_y a = -d^{-2}e(\mathcal{D}_x c)\partial_n f$ and $\partial_{r+n}\mathcal{D}_s a = d^{-2}e(\mathcal{D}_x b)\partial_n f$, where $e = (\partial_r a)[(\mathcal{D}_x b)\mathcal{D}_t c - (\mathcal{D}_x c)\mathcal{D}_t b] + (\partial_r b)[(\mathcal{D}_x c)\mathcal{D}_t a - (\mathcal{D}_x a)\mathcal{D}_t c] + (\partial_r c)[(\mathcal{D}_x a)\mathcal{D}_t b - (\mathcal{D}_x b)\mathcal{D}_t a]$. Let $e\mathcal{D}_x c \neq 0$, i.e. $\partial_{r+n}\mathcal{D}_y a \neq 0$, and show contradictoriness of (4) due to $\partial_{r+mn}\mathcal{D}_y^m a \neq 0$ and $\partial_{r+mn}\mathcal{D}_s a = 0$. Let $\mathcal{D}_x c \neq 0$ and $e = 0$, and derive from (4) that $\partial_k \mathcal{D}_y^m a = 0$ $(k \geqslant r + n)$ and $\partial_k \mathcal{D}_y^l a = 0$ $(k > r, l = 1, \ldots, m-1)$; take the system of identities $e = 0$, $\partial_k \mathcal{D}_y a = 0$ $(k > r)$ and $\partial_{r+n}\mathcal{D}_y^2 a = 0$, and show that the system is compatible only if a, b and c are functionally dependent. Consider the last possibility: $\mathcal{D}_x c = 0$, i.e. $c = c(t)$. Let $e \neq 0$, and find the orders (in u_k) of the left- and right-hand sides of (4) to be $r + n$ and $r + m$ respectively, i.e. $m = n$. Let $e = 0$ (the case of so-called degenerate MTs [3]), and find the same orders to be $r + n - 1$ and $r + m - 1$, i.e. $m = n$. □

Since the order r was not fixed, the proven theorem is valid for MTs of any order as well as for point and contact transformations. The necessary condition $c=c(t)$ simplifies a search for admissible MTs essentially. The change of time $t\to s$: $s=c(t)$ is applicable to any EE and needs no further consideration. Therefore we put $c=t$ in (1), and condition (4) takes the form

$$\mathscr{D}_t a-(\mathscr{D}_x b)^{-1}(\mathscr{D}_x a)\mathscr{D}_t b=g(t, b, a, (\mathscr{D}_x b)^{-1}\mathscr{D}_x a, \ldots, [(\mathscr{D}_x b)^{-1}\mathscr{D}_x]^n a). \tag{5}$$

Since variables $t, x, u, \ldots, u_{r+n}$ must be considered as independent in (5), we can differentiate (5) with respect to these variables and thus get new (only necessary) conditions which simplify the analysis of the necessary and sufficient condition (5). Taking ∂_{r+n} of (5) if $e=(\partial_r a)\mathscr{D}_x b-(\partial_r b)\mathscr{D}_x a\neq 0$ (non-degenerate MTs), or taking ∂_{r+n-1} of (5) if $e=0$ (degenerate MTs), we find

$$(\mathscr{D}_x b)^n F(t, x, u, u_1, \ldots)=G(t, b, a, (\mathscr{D}_x b)^{-1}\mathscr{D}_x a, \ldots) \tag{6}$$

where F and G are the separants of EEs (2) and (3) respectively, i.e $F(t, x, u, \ldots)=\partial f/\partial u_n$ and $G(t, y, v, \ldots)=\partial g/\partial v_n$. Necessary condition (6) is very informative. For example, let both EEs (2) and (3) have constant separants. In this case, $\mathscr{D}_x b=\alpha=$ constant $\neq 0$ due to (6), i.e. $y=\alpha x+\sigma(t)$. Up to such a transformation of x, every MT between two constant separant EEs is $v=a(t, x, u, \ldots, u_r)$, $y=x$, $s=t$, and the necessary and sufficient condition (5) takes the form

$$\mathscr{D}_t a=g(t, x, a, \mathscr{D}_x a, \ldots, \mathscr{D}_x^n a). \tag{7}$$

Namely, this kind of MTs has numerous applications. Owing to the simplicity of (7), even a classification of such MTs is possible for low-order EEs [15]. However, more general MTs with $\mathscr{D}_x b\neq$ constant are required, when one intends to connect a constant separant EE with a non-constant separant EE [2, 7, 14]. The following theorem describes the structure of MTs in a very general case, when the only restriction $F=1$ is imposed on EE (2).

Theorem 2. Every MT (1), admitted by EE (2) of the form $u_t=u_n+\tilde{f}(t, x, u, \ldots, u_{n-1})$, consists of the following three transformations: (i) $(t, x, u)\to(t, x, w(t, x))$, where $w=\tilde{a}(t, x, u, \ldots, u_{x\ldots x})$; (ii) $(t, x, w)\to(t, y, v(t, y))$, where $y=x$ and $v=w$, or $y=w$ and $v=w_x$, or $y=w_x$ and $v=w_{xx}$; (iii) a transformation of time $s=c(t)$ and a contact transformation of (y, v).

Sketch of proof. Assume $\partial_r a\neq 0$ without loss of generality. Put $F=1$, and get from (6) that the maximal order of G (in v_k) is 2 for degenerate MTs and 1 for non-degenerate MTs. In the degenerate case, represent a as $a=h(t, x, u, \ldots, u_{r-1}, b)$, $\partial_x h+\Sigma_{k=0}^{r-1} u_{k+1}\partial_k h=0$, and get from (6) that $\partial^2 h/\partial b^2$ is a function of t, b, h and $\partial h/\partial b$ only. Prove that a contact transformation of (y, v) exists which changes the degenerate MT to a non-degenerate MT of order $r-1$. Consider the non-degenerate case. If $G=G(t, y)$, make $G=1$ by a transformation of y, and then take $b=x$ (without loss of generality) due to (6). If $G=G(t, y, v)$ or $G=G(t, y, v, v_1)$, make $G=v^n$ by point or contact transformations of (y, v), respectively, and then take $a=\mathscr{D}_x b$ due to (6). (The subsequent part of proof is independent of the restriction $F=1$.) Consider the MT $s=t$, $y=b(t, x, u, \ldots, u_{r-1})$, $v=\mathscr{D}_x b$. Substitute $a=\mathscr{D}_x b$ to (5), and prove the existence of $q(t, x, u, \ldots, u_{r-2})$ such that $\mathscr{D}_t b-q\mathscr{D}_x b$ is a function of t, b, $\mathscr{D}_x b, \ldots, \mathscr{D}_x^n b$ only and $\mathscr{D}_x q$ is a function $h(t, b)$. Let $\partial h/\partial b=0$, and find $\mathscr{D}_t b=g'(t, b, \ldots, \mathscr{D}_x^n b)$, i.e. EE (2) admits the MT $w(t, x)=b(t, x, u, \ldots, u_{r-1})$ due to (7). Let $\partial h/\partial b\neq 0$, make $b=\mathscr{D}_x q$ by a point transformation of (y, v), and find $\mathscr{D}_t q=g''(t, q, \ldots, \mathscr{D}_x^n q)$, i.e. EE (2) admits the MT $w(t, x)=q(t, x, u, \ldots, u_{r-2})$ due to (7). □

The proven theorem indicates exceptionality of the Ibragimov transformation $y=u$, $v=u_x$ and the unnamed transformation $y=u_x$, $v=u_{xx}$ which map constant separant EEs $u_t=u_n+\ldots$ to non-constant separant EEs $v_t=v^n v_n+\ldots$. For convenience of application, we can find from (5) the following complete classes of EEs connected by these remarkable MTs:

$$u_t=\alpha u_1 x+u_1 p(t,u,u_1,u_1^{-1},u_2,\ldots,(u_1^{-1}\mathscr{D}_x)^{n-1}u_1)$$
$$y=u \qquad v=u_1 \tag{8}$$
$$v_t=\alpha v+v^2\mathscr{D}_y p(t,y,v,v_1,\ldots,v_{n-1})$$

$$u_t=(\alpha u_1+\beta)x+(\gamma u_1+\delta)u+p(t,u_1,u_2,u_2^{-1}u_3,\ldots,(u_2^{-1}\mathscr{D}_x)^{n-2}u_2)$$
$$y=u_1 \qquad v=u_2 \tag{9}$$
$$v_t=(2\alpha+\delta+3\gamma y)v-[\beta+(\alpha+\delta)y+\gamma y^2]v_1+v^2\mathscr{D}_y^2 p(t,y,v,v_1,\ldots,v_{n-2})$$

where functions p, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ and order n are arbitrary, $\mathscr{D}_y=\partial/\partial y+\Sigma_{k=0}^{\infty}v_{k+1}\partial/\partial v_k$. Now, let us see how theorems 1 and 2, condition (7) and classes (8) and (9) work.

Example. Let EE (2) be the Korteweg-de Vries equation (KdV) $u_t=u_3+uu_1$. What are admissible MTs (1) and resultant EEs (3) in this case? (Note the difference: here the KdV itself is mapped to other EEs, whereas the original MT [1] maps the modified KdV to the KdV.) The order of the KdV is 3, therefore theorem 1 demands $c=c(t)$ and $m=3$, and we may take $s=t$. The separant of the KdV is 1, therefore we can use theorem 2 instead of direct analysis of condition (5). According to point (i) of theorem 2, we *need to find all* MTs *of the form* $w(t,x)=a(t,x,u,\ldots,u_r)$ first. We have $G=1$ due to (6), then ∂_{r+2} and ∂_{r+1} of (7) give us a system of two conditions which is incompatible at $r>0$. Thus, only point transformations $w=a(t,x,u)$ are admissible. Proceeding to point (ii) of theorem 2, we make the transformation $w=a(t,x,u)$ of the KdV, compare the obtained EE $w_t=w_{xxx}+\ldots$ with the classes $u_t=\ldots$ of (8) and (9), and find that the KdV admits the MT $y=w$, $v=w_x$ after $w=\varphi(t,u+\lambda t^{-1}x)$, where φ is arbitrary and $\lambda=0$ or 1, and the MT $y=w_x$, $v=w_{xx}$ after $w=\mu(t)u+\nu(t)x+\rho(t)$, where μ, ν and ρ are arbitrary. The resultant EEs $v_t=\ldots$ of (8) and (9) contain these arbitrary functions too, but we eliminate the arbitrariness by point transformations of (y,v); we make also a time transformation in the case $\lambda=1$ for removing t from the right-hand side of the resultant EE. Consequently, up to arbitrary contact transformations of (y,v) and time transformations $s\to c(s)$, there are only three EEs which the KdV $u_t=u_3+uu_1$ can be mapped to by MTs, namely:

$$s=t \qquad y=u \qquad v=u_1$$
$$v_s=v^3v_3+3v^2v_1v_2+v^2 \tag{10}$$

$$s=-\tfrac{1}{3}\ln(\tfrac{1}{9}t) \qquad y=(3t)^{2/3}(u+t^{-1}x) \qquad v=-3t(u_1+t^{-1})$$
$$v_s=v^3v_3+3v^2v_1v_2+v^2-yv_1+3v \tag{11}$$

$$s=t \qquad y=u_1 \qquad v=u_2$$
$$v_s=v^3v_3+3v^2v_1v_2-y^2v_1+3yv. \tag{12}$$

In conclusion, let us focus our attention on the obtained EEs (10)-(12) and consider them from the standpoint of so-called integrability criteria. EEs (10)-(12) fail to pass the Painlevé test formulated in [16]. EE (10) has an infinite algebra of generalized symmetries with the recursion operator $R = v^2\mathscr{D}_y^2 + vv_1\mathscr{D}_y + 2vv_2 + \frac{2}{3}y + (v^3v_3 + 3v^2v_1v_2 + v^2)\mathscr{D}_y^{-1}v^{-2}$, therefore (10) has an infinite set of conserved densities which can be derived from the symmetries (consult e.g. [3] for the technique). As for EEs (11) and (12), they have neither non-trivial algebra of generalized symmetries nor infinite set of conserved densities. Indeed, EEs (11) and (12) have the separant $G = v^3$, but the quantity $G^{-1/n} = v^{-1}$ is not a conserved density for them; therefore these EEs can possess neither one generalized symmetry of order k, $k > 3$ [2] nor two conserved densities with the characteristics of orders k and l, $k > l > 4$ [17]. Thus, being nothing but the KdV transformed, EEs (11) and (12) look like non-integrable equations from the standpoint of best integrability criteria! Therefore, facing an equation which seems hopeless, but having the technique of MTs, we must not lose hope.

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